TRIGONOMETRY

Graphing the Trigonometric Functions

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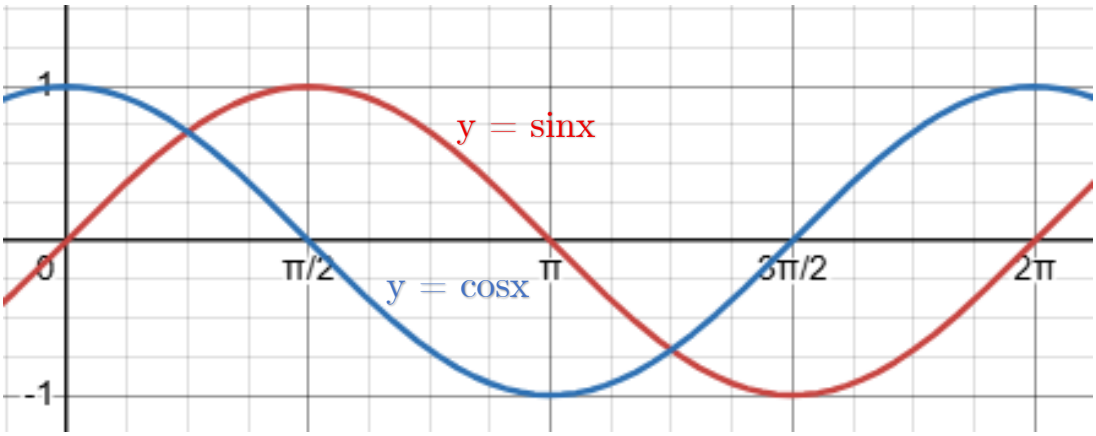
Each trigonometric function can be represented on a graph. Out of the six trigonometric functions, the functions tan, cot, sec and csc are undefined at certain values of θ because they all are expressed as ratios with denominators of sin or cos, which can be 0. This will become clear once we represent them on graphs. Degrees can be used but here, we have used radians.

Sin and Cos



When on a graph, the trig functions of sine and cosine produce 'sine waves'. So understand, both sin and cos produce sine waves; the difference between them is that they are shifted by $\frac{\pi}{2}$ or 90°. Thus, cosine is the complement of sine.

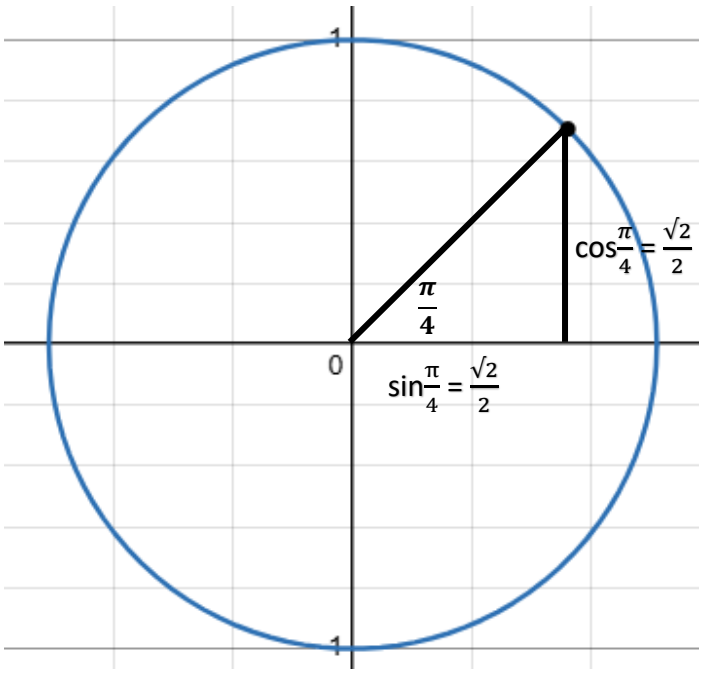
Now there are several key takeaways from this graph. First is that sin and cos extend infinitely but never exceed 1 or -1. And so, their range is from -1 to 1. We will discuss range in detail later. Also note, sine starts at 0, the origin, while cosine starts at 1. Notice that, when both sine and cosine are represented on a graph, there are several points where they intersect. These points are at intervals of π from $\frac{\pi}{4}$ where they have the same sign and same value of M, $\frac{\sqrt{2}}{2}$.

A period is the length of a function's cycle. In other words, it is the distance on the graph's horizontal axis that it takes for the function to repeat itself. Here we have included exactly one of sine and cosine, which is, 2π rad. The period can also be considered as the peak-to-peak distance between the function's peaks.

Tan and Cot

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad\qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

Both tan and cot have a denominator of cos or sin respectively. Since both sin and cos can be zero at certain angles, for example, $\sin 0 = 0$ or $\cos\frac{\pi}{2} = 0$, tan and cot are undefined at those angles as dividing by zero is undefined. Firstly, let us look at the tan function.

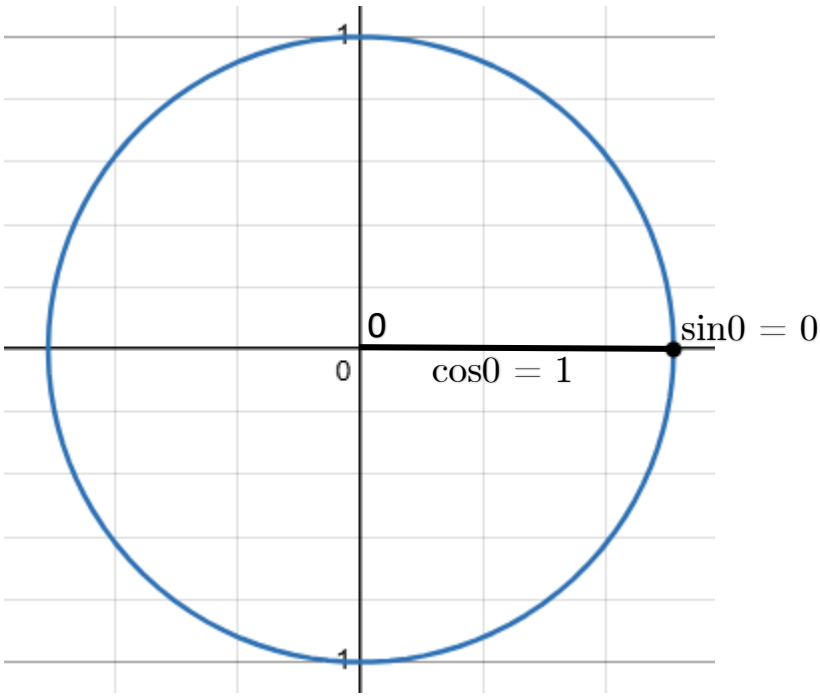


When $\theta = \frac{\pi}{4}$,

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$\tan\frac{\pi}{4} = \frac{\sqrt{2}/2}{\sqrt{2}/2}$$

$$\tan\frac{\pi}{4} = 1$$



When $\theta = 0$,

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$\tan 0 = \frac{0}{1}$$

$$\tan\frac{\pi}{4} = 0$$

Thus, tan is 0 at 0 rad and 1 at $\frac{\pi}{4}$. Now, when we approach the quadrantal angle of $\frac{\pi}{2}$, we occur a paradox. From Quadrant I, sin approaches 1, and cos approaches 0 and so tan approaches $\frac{1}{0}$. With values of θ closer to $\frac{\pi}{2}$, tan becomes a very large number as the denominator, sinθ, becomes very small. However, coming from the II Quadrant, cosine is negative so tan approaches a very large negative number.

$$\tan 89^\circ = 57.28...$$

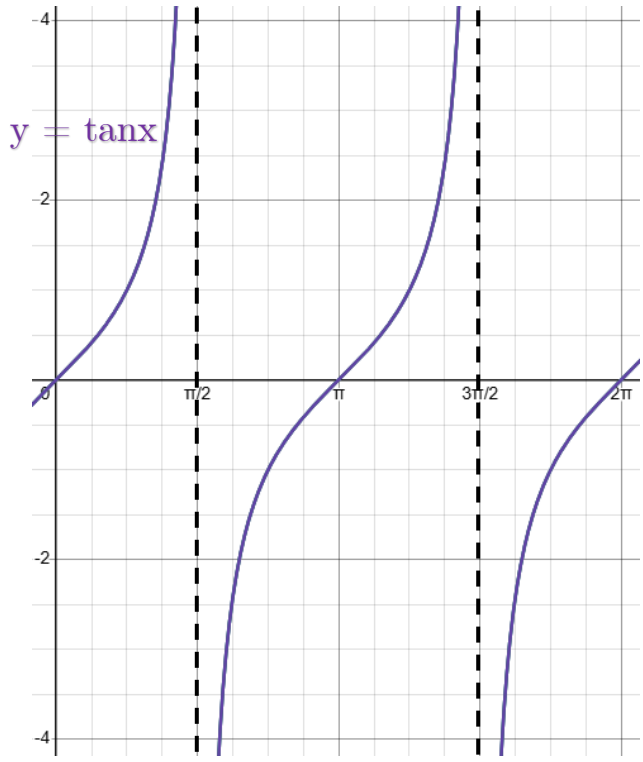
$$\tan 89.999^\circ = 57295.77...$$

$$\tan 91^\circ = -57.28...$$

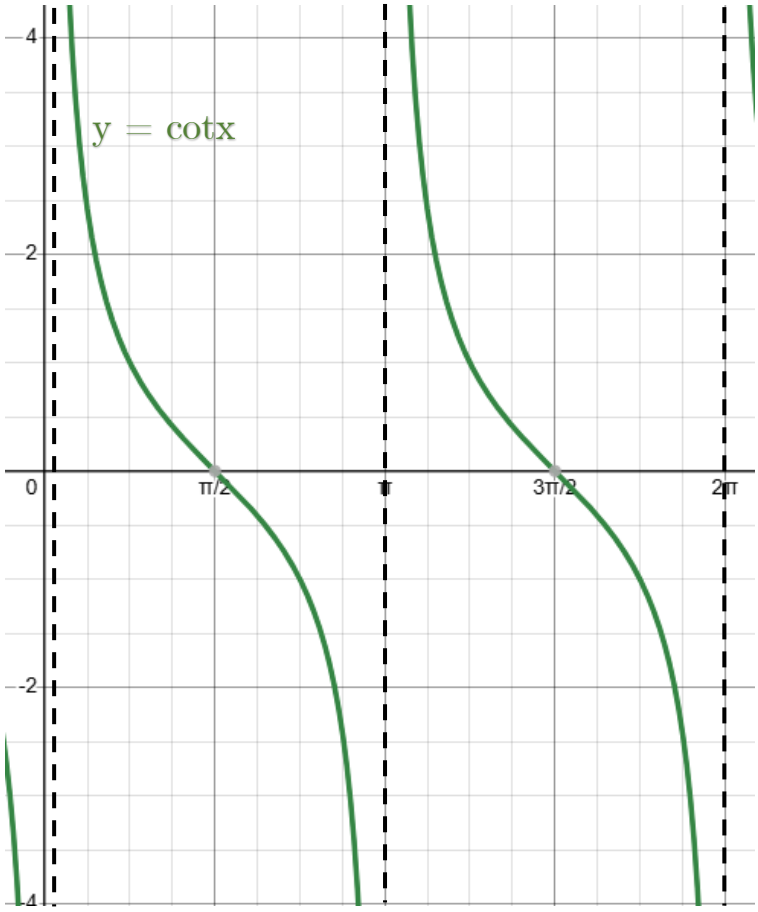
$$\tan 90.001^\circ = -57295.77...$$

Thus, as tan approaches $\frac{\pi}{2}$, it simultaneously approaches positive infinity from Quadrant I and negative infinity from Quadrant II, so $\tan\frac{\pi}{2}$ or $\tan 90^\circ$, is undefined.

When represented on a graph, the tan function approaches infinity and comes back from negative infinity once 'crossing' the "asymptote". An asymptote is a boundary that a graphed curve can approach but not cross; it is often represented as a dotted line. But remember, that, any function can never cross an asymptote. As $\tan 0 = 0$, the tan function starts at the origin and reaches 1 at $\tan\frac{\pi}{4}$ or $\tan 45^\circ = 1$, before continuing to infinity.



For cotangent, everything discussed is the same except cosine is the denominator, and the function's graph is shifted by 90° or $\frac{\pi}{2}$rad since cotangent is tangent's complement. Like tangent, it also produces periodic asymptotes; unlike tangent, cotangent starts at positive infinity, i.e., it is undefined at 0 where sin is 0.



Sec and Csc

Secant and Cosecant both have numerators of 1 and denominators of cosine and sine respectively. Thus, they will each have vertical asymptotes where sin or cos is 0.

First, we will begin with Secant.

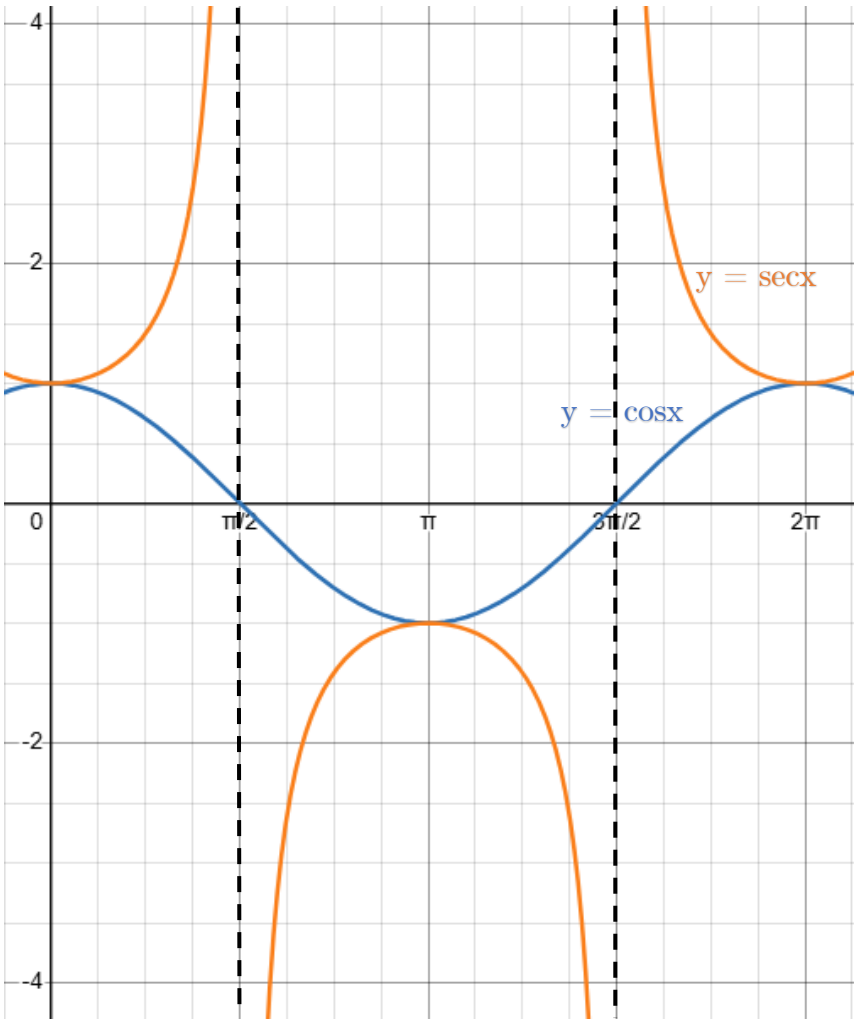
$$\sec\theta = \frac{1}{\cos\theta}$$

At π multiples of θ starting from $\frac{\pi}{2}$, cos = 0. So $\frac{1}{\cos\theta}$ will be undefined at those points. Similarly to tangent, approaching secant of θ from Quadrant I gives a very large positive number as cosine becomes smaller, but from Quadrant II, secant becomes a very large negative number. Thus, like tangent, vertical asymptotes occur at $\frac{\pi}{2}$, $\frac{3\pi}{2}$, $\frac{5\pi}{2}$,

etc., where secant appears to approach positive infinity and come from negative infinity.

secθ starts at 1 because when θ = 0, cos0 = 1 and sec0 = $\frac{1}{1}$ = 1, before going to infinity at $\frac{\pi}{2}$ because cos$\frac{\pi}{2}$ = 0 and sec$\frac{\pi}{2}$ = $\frac{1}{cos\frac{\pi}{2}}$ = $\frac{1}{0}$ which is undefined. In Quadrant II, secθ comes from negative infinity and reaches -1 at π rad because cosπ = -1 so secπ = $\frac{1}{cos\pi}$ = $\frac{1}{-1}$ = -1, before again going to negative infinity at $\frac{3\pi}{2}$ as cosθ becomes a small negative number.

Here is the graph of secθ and cosθ.

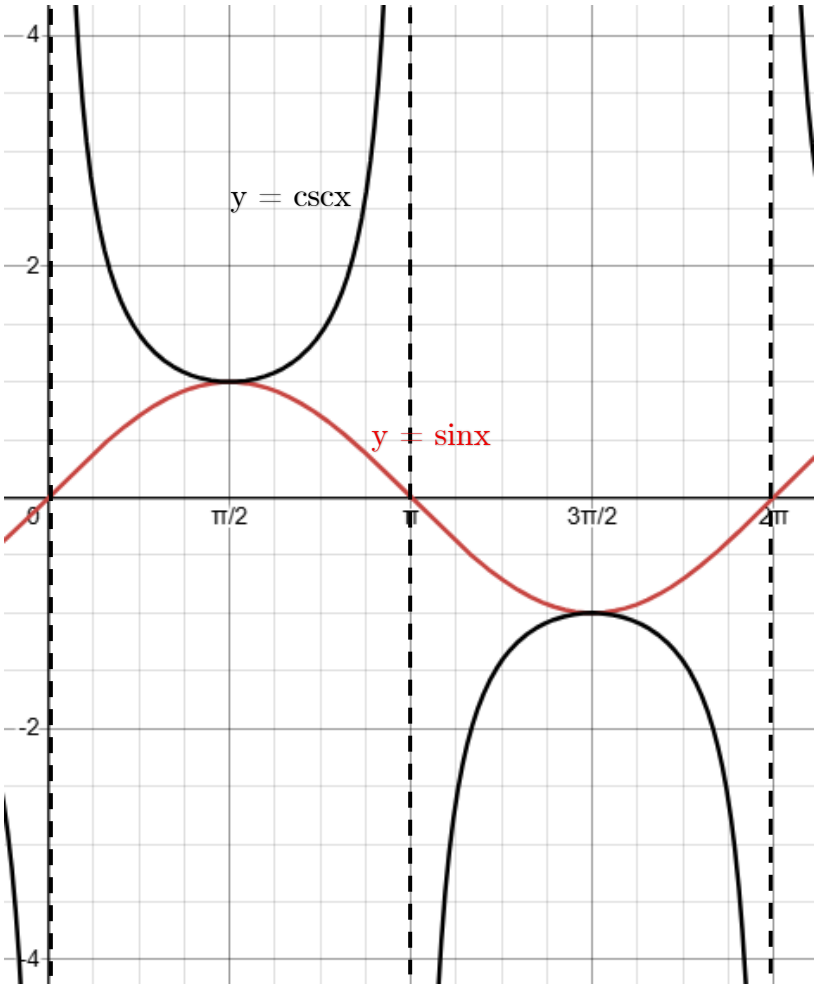


Like how cotangent is the complement of tangent, cosecant is the complement of secant and, its graph is shifted by 90° or $\frac{\pi}{2}$ rad.

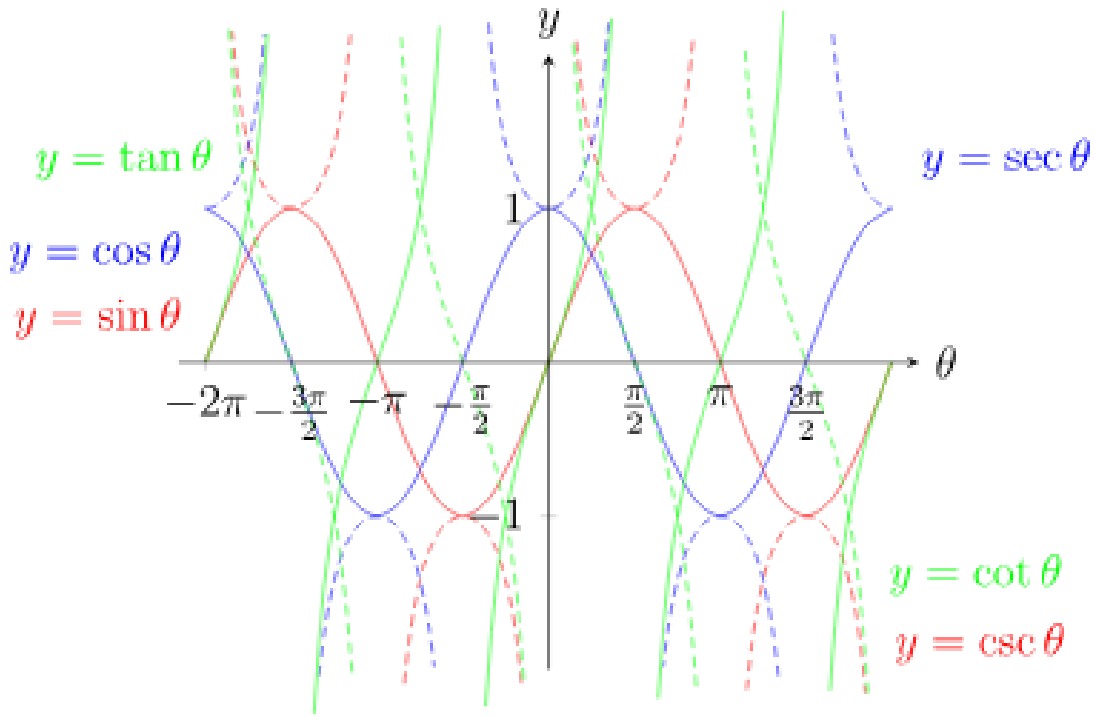
$$\csc\theta = \frac{1}{sin\theta}$$

Since, sinθ is 0 when θ = 0, csc0 = $\frac{1}{sin0}$ = $\frac{1}{0}$ which is undefined. Because of this, cosecant starts at positive infinity and is undefined at 0. What explained before for secant also applies for cosecant but sinθ is used instead of cosθ.

Here is the graph of cscθ and sinθ.



And so, with all this acquired knowledge, here is the graph with all the trigonometric functions where f(x) = y.



Domain and Range of Trig Functions

Until now, we have learnt how the trigonometric functions can be represented on graphs. However, one may notice, that functions like sin and cos can only go from -1 to 1. Logically, based on this fact, sin and cos can never return any value greater or lesser than 1 or -1. In other words, the "Range" of a function is the set of all values that the function can return as an output. In this case, the range of sin and cos is from -1 to 1.

The domain of a function is the set of all values that a function can accept as an argument (for which the function is defined). For example, cotθ is undefined at values of π, 2π, 3π, etc. (also including its negatives). Therefore, the domain of θ is any real number θ that is not nπ where n = 0, ± 1, ± 2, ± 3... In contrast, sin and cos can accept any real number θ, thus their domain is all real values of θ.

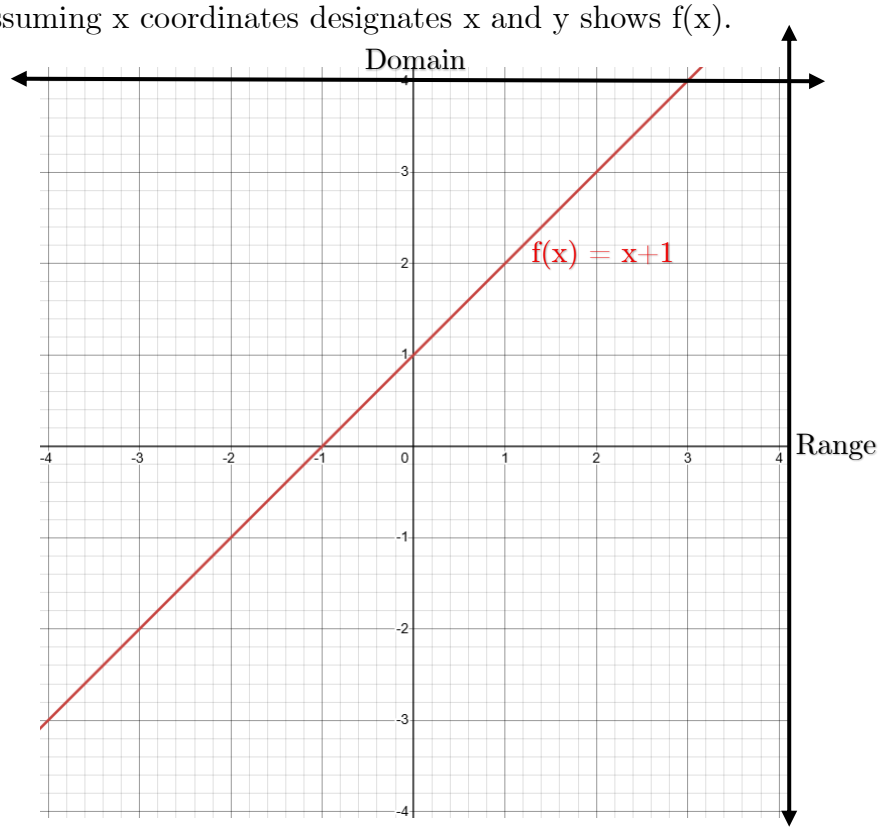
Mathematically, this is represented as —

$$\text{Range of sin and cos} = \{[-1,1]\}$$

$$\text{Domain of sin and cos} = \{\text{All real } \theta\}$$

But before we see the domain and range of all the trig functions, we need to understand the domains and ranges of any function you might encounter.

A function is usually represented in the form f(x) = y, read as 'f of x equals y'. We will take a function, f(x) = x+1, where x is the input. On a graph, the domain is how far the function stretches left to right, and range represents how far it stretches top to bottom, assuming x coordinates designates x and y shows f(x).



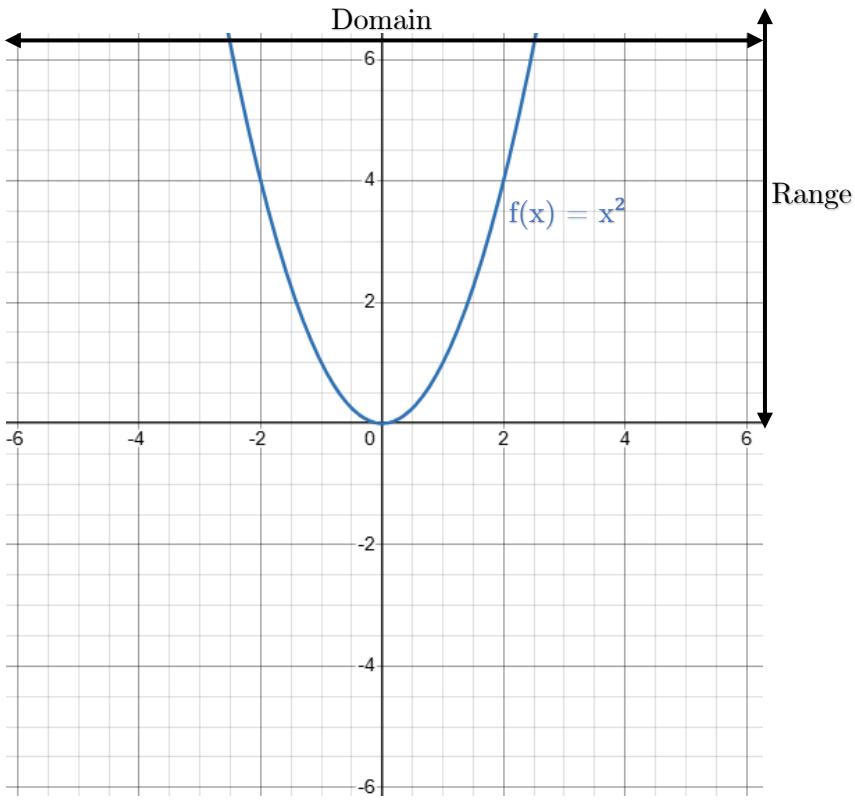
As you can see, there is no constraint to the values the function can take or even return, i.e., the domain and range spans infinitely.

$$f(x) = x+1 = y$$

Domain: {All real x} or {[-∞, ∞]}

Range: {All real y} or {[-∞, ∞]}

For another function, say, $f(x) = x^2$, the function graph is —



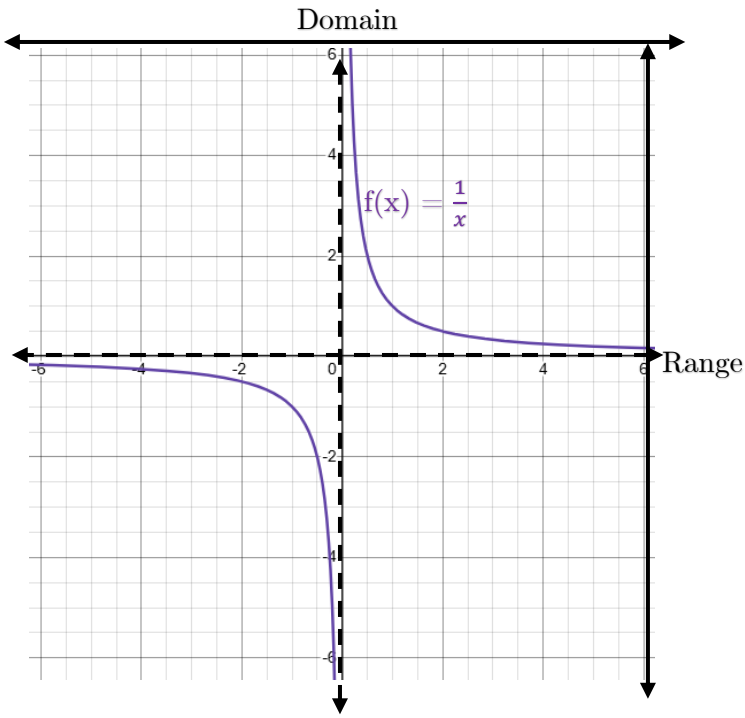
Thus, its domain and range are —

$$f(x) = x^2 = y$$

Domain: {All real x} or {[-∞, ∞]}

Range: {[0, ∞]}

And for $f(x) = \frac{1}{x}$,



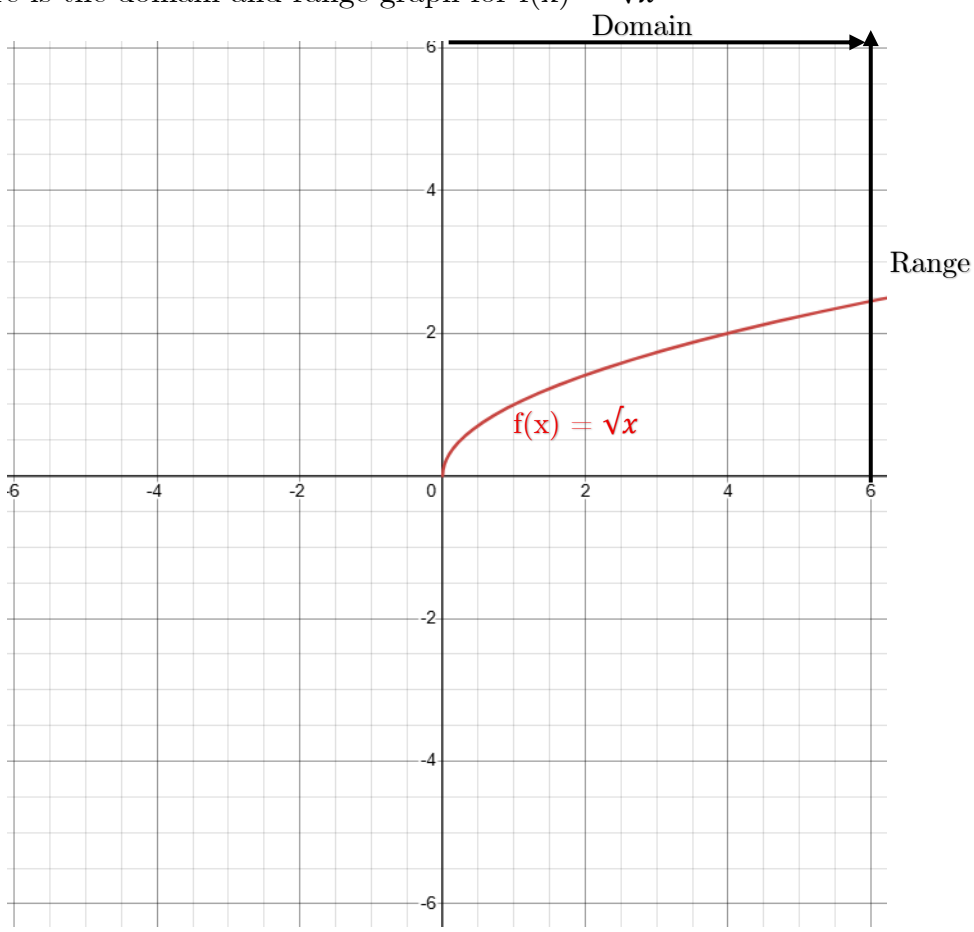
Here, the function accepts all values of x except 0 because at 0, f(0) = $\frac{1}{0}$ which is undefined for reasons explained earlier (remember the paradox when graphing tan?). Thus, in standard notation, it is —

$$f(x) = \frac{1}{x} = y$$

$$\text{Domain: } \{x : x \neq 0\} \text{ or } (-\infty, 0) \cup (0, \infty)$$

$$\text{Range: } \{y : y \neq 0\} \text{ or } (-\infty, 0) \cup (0, \infty)$$

Finally, here is the domain and range graph for f(x) = $\sqrt{x}$



The domain and range are both constrained from 0 to positive infinity. Thus, their domain and range are —

$$f(x) = \sqrt{x} = y$$

$$\text{Domain: } \{x : x \geq 0\} \text{ or } \{[0, \infty)\}$$

$$\text{Range: } \{y : y \geq 0\} \text{ or } \{[0, \infty)\}$$

As they must be greater than or equal to 0.

So now, let's finally find the domain and range of the trigonometric functions.

We know that the domain of sinθ and cosθ is {All real θ} and their range is {[-1, 1]}. Regarding cotangent, cotθ is undefined at π multiples from 0. Thus, the domain of cotθ is {θ: θ ≠ nπ}, n being 0, ±1, ±2, ±3... For tangent however, tanθ is undefined at π multiples of θ starting from $\frac{\pi}{2}$. So its domain is {θ: θ ≠ $\frac{(2n+1)\pi}{2}$} n being, 0, ±1, ±2, ±3... Since they can have any number as their output, the range of tangent and cotangent is {All real y}.

Similarly to cotangent, cosecant is also undefined at π multiples of θ from 0 (refer graph), so its domain is {θ: θ ≠ nπ}. And similarly to tangent, secant is undefined at π multiples of θ from $\frac{\pi}{2}$ and so its domain is {θ: θ ≠ $\frac{(2n+1)\pi}{2}$}. In contrast though, secant and cosecant have ranges from -1 to negative infinity and +1 to positive infinity. Usually, it is expressed as {(-∞, -1]} U {[1, ∞)}, though this may differ.

And so, here is the table of all the trig functions and their respective domains and ranges. Keep in mind, you should be able to figure out the domains and ranges not by memorising the table but by visualizing the graphs.

Function	Domain	Range
sinϑ	{All real θ}	{[-1, 1]}
cosϑ	{All real θ}	{[-1, 1]}
tanϑ	{θ: θ ≠ $\frac{(2n+1)\pi}{2}$}	{All real y}
cotϑ	{θ: θ ≠ nπ}	{All real y}
secϑ	{θ: θ ≠ $\frac{(2n+1)\pi}{2}$}	{(-∞, -1]} U {[1, ∞)}
cscϑ	{θ: θ ≠ nπ}	{(-∞, -1]} U {[1, ∞)}